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Variable Structure Control of a Class of Uncertain Systems

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ABSTRACT

This brief paper proposes a method for tuning the parameters of a variable structure controller. The approach presented extracts the error at the output of the controller and applies a nonlinear tuning law using this error measure. The adaptation mechanism drives the state tracking error vector to the sliding hypersurface and maintains the sliding mode. In the simulations, the approach presented has been tested on the control of Duffing oscillator and the analytical claims have been justified under the existence of measurement noise, uncertainty and large nonzero initial errors.

Keywords: Sliding Mode Control, Tuning Laws, Nonlinear Systems, and Robust Control

1. INTRODUCTION

Parameter tuning in adaptive control systems has been a core issue in dealing with uncertainties and imprecision. One good alternative to robustify the control system against disturbances and uncertainties is to exploit a Variable Structure Control (VSC) scheme (Hung, Gao & Hung, 1993; Utkin, 1992; Slotine & Li, 1991). The scheme is well-known with its robustness against unmodeled dynamics, disturbances, time delays and nonlinearities (Young, Utkin & Ozguner, 1999). A later trend in the field of VSC

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design is to exploit the strength of the technique in parameter tuning issues (Sira-Ramirez & Colina-Morles, 1995; Yu, Zhihong & Rahman, 1998; Parma, Braga & Menezes, 1998). The resulting system exhibits the robustness and invariance properties inherited from VSC technique. As long as the target output of the adaptive system is known, the utilization of the mentioned techniques reveals good performance. However, in control applications, the lack of a priori knowledge on the target control signal leads the designer to seek for alternative methods predicting the error on the control signal (Efe, Kaynak & Yu, 2000)

This brief paper presents a method for extracting the error on the control signal particularly for the variable structure control purpose. In the second section, we describe the proposed technique for control error calculation. Simulation studies are presented next, and the concluding remarks are given at the end of the paper.

2. PROPOSED APPROACH

Consider a nonlinear and non-autonomous system $\theta^{(r)} = f\left(\theta, \dot{\theta}, \ldots, \theta^{(r-1)}, t\right) + \tau$, where $f(.)$ is an unknown function, $\underline{\theta} = [\theta, \dot{\theta}, \ldots, \theta^{(r-1)}]^{\mathrm{T}}$ is the state vector, τ is the control input to the system and t is the time variable. Defining $\underline{\theta}_d = [\theta_d, \dot{\theta}_d, \ldots, \theta_d^{(r-1)}]^{\mathrm{T}}$ as the desired state vector and $\underline{e} = \underline{\theta} - \underline{\theta}_d$ as the error vector, one can set the sliding hypersurface as $s_p(\underline{e}) = \underline{\Lambda}^{\mathrm{T}} \underline{e}$. The VSC design framework prescribes that the entries of the vector $\underline{\Lambda}$ are the coefficients seen in the analytic expansion of $s_p = (d/dt + \lambda)^{r-1}(\theta - \theta_d)$ or more generally they are the coefficients of a Hurwitz polynomial. Here λ is a positive constant. Let V_p be a candidate Lyapunov function given as $V_p(s_p) = s_p^2/2$; if the prescribed control signal satisfies $\dot{V}_p(s_p) = -s_p \xi \operatorname{sgn}(s_p)$ with $\xi > 0$, the negative definiteness of the time derivative of the above Lyapunov function is ensured. The conventional design postulates the control sequence given as

$$\tau_{smc} = -\left(f(\underline{\theta}, t) - \theta_d^{(r)} + \Lambda_r^{-1}\left(\sum_{i=1}^{r-1} \Lambda_i e^{(i)} + \xi \operatorname{sgn}(s_p)\right)\right) \tag{1}$$

which ensures $\dot{V}_p(s_p) < 0$. More explicitly, if (1) is substituted into the system dynamics, it is seen that $\dot{s}_p = -\xi \operatorname{sgn}(s_p)$ is enforced automatically. Apparently, s_p will converge to zero in finite time, which means that the error vector is confined to the sliding manifold after some time. The behavior thereafter is convergent since it takes place in the close vicinity of the sliding manifold, i.e. the error vector converges to the origin as prescribed by the manifold equation.

Remark 1: When the control in (1) is applied to the system, we call the resulting behavior as the *target Sliding Mode Control* (SMC) and the input vector leading to it as the *target control sequence* (τ_{smc}). Since the functional form of the function f is not known, it should be obvious that τ_{smc} cannot be constructed by following the traditional SMC design approach.

Definition 2: Given the system $\theta^{(r)} = f(\underline{\theta}, t) + \tau$, and a desired trajectory $\underline{\theta}_d(t)$ for $t \geq 0$, the input sequence satisfying the differential equation $\theta_d^{(r)} = f(\underline{\theta}_d, t) + \tau_d$ is defined to be the *idealized control sequence* denoted by τ_d, and the differential equation itself is defined to be the *reference SMC model*. Mathematically, the existence of such a model and the sequence means that the system perfectly follows the desired trajectory if both the idealized control sequence is known and the initial conditions are set as $\underline{\theta}(t=0) = \underline{\theta}_d(t=0)$, more explicitly $\underline{e}(t) \equiv 0$ for $\forall\ t \geq 0$. Undoubtedly, such an idealized control sequence will not be a norm-bounded signal when there are step-like changes in the vector of command trajectories or when the initial errors are nonzero. It is therefore that the reference SMC model is an abstraction due to the limitations of the physical reality, but the concept of idealized control sequence should be viewed as the synthesis of the command signal $\underline{\theta}_d$ from the time solution of the given differential equation.

Fact 3: If the target control sequence formulated in (1) were applied to the system, the idealized control sequence would be the steady state solution of the control signal, i.e. $\lim_{t\to\infty} \tau = \tau_d$.

Defining the control error by $s_c \triangleq \tau - \tau_d$ and rewriting the control signal with the idealized SMC model yields $\tau = \tau_d - \left(\Delta f + \Lambda_r^{-1}\left(\sum_{i=1}^{r-1}\Lambda_i e^{(i)} + \xi\,\mathrm{sgn}\left(s_p\right)\right)\right)$, where $\Delta f = f\left(\underline{\theta}, t\right) - f\left(\underline{\theta}_d, t\right)$. The target control sequence becomes identical to the idealized control sequence, i.e. $\underline{\tau} \equiv \underline{\tau}_d$, as long as the condition given below holds true.

$$\Delta f = -\Lambda_r^{-1}\left(\sum_{i=1}^{r-1}\Lambda_i e^{(i)} + \xi\,\mathrm{sgn}\left(s_p\right)\right). \tag{2}$$

However, this condition is of no practical importance as we do not have the analytic form of the function f. Therefore, one should consider this equality as an equality to be enforced instead of an equality that holds true all the time, because its implication is s_c=0, which is the aim of the design.

After straightforward manipulations, $\dot{s}_p$ can be rewritten as $\dot{s}_p = \Lambda_r\left(\Delta f + s_c\right) + \sum_{i=1}^{r-1}\Lambda_i e^{(i)}$. Inserting (2) into $\dot{s}_p$ and solving for s_c gives

$$s_c = \Lambda_r^{-1}\left(\dot{s}_p + \xi\,\mathrm{sgn}\left(s_p\right)\right). \tag{3}$$

Remark 4: It should be noted that the application of τ_d to the system with zero initial errors would lead to $\underline{e}(t) \equiv \underline{0}$ for $\forall\ t \geq 0$, however, τ_d is not a computable quantity. On the other hand, the application of τ_{smc} to the system will lead to s_p=0 for $\forall\ t \geq t_h$, where t_h is the hitting time, and the origin would be reached according to the dynamics described by the sliding manifold, but knowing τ_{smc} implies the availability of the function f(.). If one analyzes (3), a control signal minimizing the magnitude of s_c would introduce all trajectories in the error space to tend to the sliding manifold, i.e. (2) is enforced without knowing the description of the function f(.) explicitly. Consequently, the tendency of such a control scheme would be to generate the target SMC sequence of (1) by utilizing the computable quantities.

Now consider the feedback control loop illustrated in Figure 1, and define the Lyapunov function $V_c(s_c)=s_c^2/2$, which is a measure of how well the controller performs.

Remark 5: An adaptation algorithm ensuring $\dot{V}_c(s_c)<0$ when $s_c\neq 0$ enforces (2) to hold true and creates the predefined sliding regime after a reaching mode lasting until the hitting time denoted by t_h, beyond which s_c=0 as the system is in the sliding regime.

Consider the controller $\tau=\underline{\phi}^{\mathrm{T}}\underline{u}$, where $\underline{\phi}$ is the vector of adjustable parameters and $\underline{u}=[\underline{e}^{\mathrm{T}} \quad 1]^{\mathrm{T}}$. Choose the following Lyapunov function candidate:

$$V_A=\mu V_c+\rho\frac{1}{2}\left\|\frac{\partial V_c}{\partial\underline{\phi}}\right\|^2 \tag{4}$$

where, ||•|| is the Euclidean norm and, μ and ρ are positive constants determining the relative importance of the terms.

Remark 6. A likely question that can be raised at this point would be how such a Lyapunov function is selected. After straightforward manipulations, it can be shown that $V_A=\alpha(t)V_c$, where $\alpha(t)=\mu+\rho\underline{u}^{\mathrm{T}}\underline{u}$, or equivalently, $\alpha(t)=\mu+\rho+\rho\underline{e}^{\mathrm{T}}\underline{e}$. Referring to Figure 2, which visualizes V_A for μ=1 and ρ=10, a direct conclusion would be the fact that as $\|\underline{e}\|$ increases, the two flaps become steep, and as $\|\underline{e}\|$ decreases the local property of the surface gets shallower. Choosing such a Lyapunov function will therefore enable us to represent how well the controller performs as well as how well the plant performs jointly. As seen from the contour plot of Figure 2, the surface is symmetric with respect to s_c=0 line, and the cost of any disturbance leading to an increment in $\|\underline{e}\|$ will be more than the identical disturbance arising around s_c=0 and $\|\underline{e}\|$=0. This is particularly important since the tuning activity will be trying to cope with noise, which is substantially effective during the sliding mode, i.e. when s_c=0 is reached.

In order not to violate the constraints of the physical reality, the following bound conditions are imposed: $\|\underline{\phi}\| \le B_{\phi}$, $\|\underline{u}\| \le B_u$, $\|\dot{\underline{u}}\| \le B_{\dot{u}}$, $|\tau| \le B_{\tau}$, $|\tau_d| \le B_{\tau_d}$ and $|\dot{\tau}_d| \le B_{\dot{\tau}_d}$.

Theorem 7: If the adaptation strategy for the adjustable parameters of the controller is chosen as

$$\dot{\underline{\phi}} = -K\left(\mu I + \rho \underline{u}\underline{u}^{\mathrm{T}}\right)^{-1} \underline{u}\,\mathrm{sgn}(s_c) \tag{5}$$

with K is a sufficiently large constant satisfying $K > \left(\mu + \rho B_u^2\right)\left(B_{\phi} B_{\dot{u}} + B_{\dot{\tau}_d}\right) + \rho\left(B_{\tau} + B_{\tau_d}\right) B_u B_{\dot{u}}$; then the negative definiteness of the time derivative of the augmented Lyapunov function in (4) is ensured.

*Proof***:** Evaluating the time derivative of the Lyapunov function in (4) yields

$$\dot{V}_A = \mu\left(\left(\frac{\partial V_c}{\partial \underline{\phi}}\right)^{\mathrm{T}} \dot{\underline{\phi}} + \left(\frac{\partial V_c}{\partial \underline{u}}\right)^{\mathrm{T}} \dot{\underline{u}} + \frac{\partial V_c}{\partial \tau_d}\dot{\tau}_d\right) + \rho\left(\frac{\partial V_c}{\partial \underline{\phi}}\right)^{\mathrm{T}}\left(\frac{\partial^2 V_c}{\partial \underline{\phi} \partial \underline{\phi}^{\mathrm{T}}}\dot{\underline{\phi}} + \frac{\partial^2 V_c}{\partial \underline{\phi} \partial \underline{u}^{\mathrm{T}}}\dot{\underline{u}} + \frac{\partial^2 V_c}{\partial \underline{\phi} \partial \tau_d}\dot{\tau}_d\right). \tag{6}$$

Since the controller is $\tau = \underline{\phi}^{\mathrm{T}}\underline{u}$ and $s_c \triangleq \tau - \tau_d$, following terms can be calculated: $\left(\partial V_c / \partial \underline{\phi}\right)^{\mathrm{T}} = s_c \underline{u}^{\mathrm{T}}$, $\left(\partial V_c / \partial \underline{u}\right)^{\mathrm{T}} = s_c \underline{\phi}^{\mathrm{T}}$, $\partial V_c / \partial \tau_d = -s_c$, $\partial^2 V_c / \partial \underline{\phi} \partial \underline{\phi}^{\mathrm{T}} = \underline{u}\underline{u}^{\mathrm{T}}$, $\partial^2 V_c / \partial \underline{\phi} \partial \underline{u}^{\mathrm{T}} = \underline{u}\underline{\phi}^{\mathrm{T}} + s_c I$, and $\partial^2 V_c / \partial \underline{\phi} \partial \tau_d = -\underline{u}$. The time derivative in (6) can now be rearranged as follows;

$$\begin{aligned}
\dot{V}_A &= s_c \underline{u}^{\mathrm{T}}\left(\mu I + \rho\, \underline{u}\underline{u}^{\mathrm{T}}\right)\dot{\underline{\phi}} + s_c\left(\mu + \rho \underline{u}^{\mathrm{T}}\underline{u}\right)\left(\underline{\phi}^{\mathrm{T}}\dot{\underline{u}} - \dot{\tau}_d\right) + \rho\, s_c^2 \underline{u}^{\mathrm{T}}\dot{\underline{u}} \\
&= -K|s_c|\underline{u}^{\mathrm{T}}\underline{u} + s_c\left(\mu + \rho \underline{u}^{\mathrm{T}}\underline{u}\right)\left(\underline{\phi}^{\mathrm{T}}\dot{\underline{u}} - \dot{\tau}_d\right) + \rho\, s_c^2 \underline{u}^{\mathrm{T}}\dot{\underline{u}} \\
&\le -K|s_c| + s_c\left(\mu + \rho \underline{u}^{\mathrm{T}}\underline{u}\right)\left(\underline{\phi}^{\mathrm{T}}\dot{\underline{u}} - \dot{\tau}_d\right) + \rho\, s_c^2 \underline{u}^{\mathrm{T}}\dot{\underline{u}} \qquad (7) \\
&\le -K|s_c| + |s_c|\left(\mu + \rho B_u^2\right)\left(B_{\phi} B_{\dot{u}} + B_{\dot{\tau}_d}\right) + \rho\, s_c^2 \underline{u}^{\mathrm{T}}\dot{\underline{u}} \\
&\le -K|s_c| + |s_c|\left(\mu + \rho B_u^2\right)\left(B_{\phi} B_{\dot{u}} + B_{\dot{\tau}_d}\right) + \rho\, |s_c|\left(B_{\tau} + B_{\tau_d}\right) B_u B_{\dot{u}}
\end{aligned}$$

The last inequality above is due to the fact that $s_c^2 = |s_c|(\tau - \tau_d) \le |s_c|\left(B_\tau + B_{\tau_d}\right)$. The selection of the parameter K ensures the negative definiteness of the time derivative of the Lyapunov function in (4) and proves Theorem 7. □

Since $\left(\mu I + \rho \underline{u}\underline{u}^{\mathrm{T}}\right)^{-1} = \frac{1}{\mu} I - \frac{\rho \underline{u}\underline{u}^{\mathrm{T}}}{\mu\left(\mu + \rho \underline{u}^{\mathrm{T}}\underline{u}\right)}$, the tuning law of (5) can be paraphrased as $\dot{\underline{\phi}} = -K \frac{\underline{u}}{\mu + \rho \underline{u}^{\mathrm{T}}\underline{u}} \operatorname{sgn}(s_c)$. Apparently if $\underline{e}^{\mathrm{T}}\underline{e} \le \varepsilon$ holds true, where $\varepsilon>0$, the first r entries of the parameter vector will dominantly be influenced by the noise terms (η_i) corrupting the state vector. More explicitly, $\theta^{(i)} \cong \theta_d^{(i)}$ and $\dot{\phi}_i = -K \frac{\theta^{(i)} - \theta_d^{(i)} + \eta_i}{\mu + \rho + \rho \sum_{j=1}^{r}\left(\theta^{(j)} - \theta_d^{(j)}\right)^2} \operatorname{sgn}(s_c) \approx -K \frac{\eta_i}{\mu + \rho} \operatorname{sgn}(s_c)$ with $i=1,\ldots,r$. However, the $(r+1)^{\text{th}}$ entry of the parameter vector will be tuned by $\dot{\phi}_{r+1} = -K \frac{1}{\mu + \rho \underline{u}^{\mathrm{T}}\underline{u}} \operatorname{sgn}(s_c)$. Therefore, once $\underline{e}^{\mathrm{T}}\underline{e} \le \varepsilon$ is satisfied, the tuning of the first r parameters are stopped and only the $(r+1)^{\text{th}}$ entry is tuned. If $\underline{e}^{\mathrm{T}}\underline{e} > \varepsilon$, all adjustable parameters are tuned. This mechanism ensures that the parameter tuning due to the noise sequence is suppressed in the vicinity of the origin. Since K is designed for the worst possible conditions, the time derivative in (7) will always be negative.

Remark 8. Given system of structure $\theta^{(r)} = f(\underline{\theta}, t) + \tau$, where the function f is unknown, and a desired trajectory $\underline{\theta}_d(t)$, assuming that the SMC task is achievable, utilization of (3) as the control error together with the tuning law of (5) for the controller $\tau = \underline{\phi}^{\mathrm{T}}\underline{u}$ enforces the desired reaching mode followed by the sliding regime for some set of design parameters μ, ρ, ξ and $\underline{\Lambda}$.

3. SIMULATION STUDY

In the simulations, we test the performance of the proposed scheme on the control of a Duffing oscillator described by the following differential equation;

$$\ddot{\theta} = -p_1\theta - p_2\theta^3 - p\dot{\theta} + q\cos(\omega_d t) + \tau \qquad (9)$$

where, p_1 = 1.1, p_2 = 1, p = 0.4, q = 2.1 and ω_d = 1.8. The control problem is to enforce the states to the periodic orbit described as $\ddot{\theta}_d = \sin(\theta_d)$ with $\theta_d(0) = 1$ and $\dot{\theta}_d(0) = 0$. The identification and control of the system in (9) have previously been studied by Poznyak, Yu & Sanchez (1999). It must be noted that the enforced trajectory is radically different from the stable limit cycle of the system dynamics, and this fact requires continuous control effort.

In the simulation results presented, we set μ=1, ρ=10 and $\underline{\Lambda}=[1\ 1]^T$, ξ=1, K=1000 and ε=0.001. The block diagram of the control system is depicted in Figure 1 in detail. The measurement noise sequences for both states are Gaussian distributed, zero mean and both have equal standard deviations, which is 0.0025. The disturbance caused by the measurement noise satisfies $|\eta_i(t)| \le 0.001$ with probability very close to unity.

In Figure 3, the phase space behavior for $\theta(0)=-1$ and $\dot{\theta}(0)=0$ have been demonstrated. The plot seen figures out that $\dot{e} = -e$ (λ=1 or s_p=0) line is the attracting invariant. Clearly the error vector is guided towards the sliding manifold and due to the design presented, it is forced to remain in the vicinity of the attracting loci without explicitly knowing the analytical details of the function *f*. However, it can fairly be claimed that the sliding manifold is most probably a locally invariant subspace as the results heavily depend upon the unknown function *f*.

In Figure 4, the applied control signal and the evolution of the controller parameters are illustrated. Although the exact use of the sgn(.) function in (3) introduces some amount of high frequency components, the produced control sequence is sufficiently smooth and reasonable in magnitude. The evolution of the controller parameters ($\underline{\phi}=[\phi_1\ \phi_2\ \phi_3]^T$) is apparently bounded as seen in the figure.

Finally, the presented technique is computationally inexpensive, for the considered application, the total number of floating point operations for the control calculation and tuning is equal to 36 with 2 comparisons for sign function evaluations. This result stipulates that the computational complexity of the presented technique is affordable even for low speed microprocessors.

4. CONCLUSIONS

This brief paper introduces a novel approach for creating and maintaining the sliding motion in the behavior of an uncertain system. The system under control is of unknown structure and it is under the ordinary feedback loop with an adaptive variable structure controller. The presented results have demonstrated that the predefined sliding regime could be created and maintained if the controller parameters are tuned in such a way that the reaching is enforced. Computational simplicity of the method is another prominent feature that should be emphasized.

Future research aims to discover the properties of the class of functions determining the applicability range of the approach.

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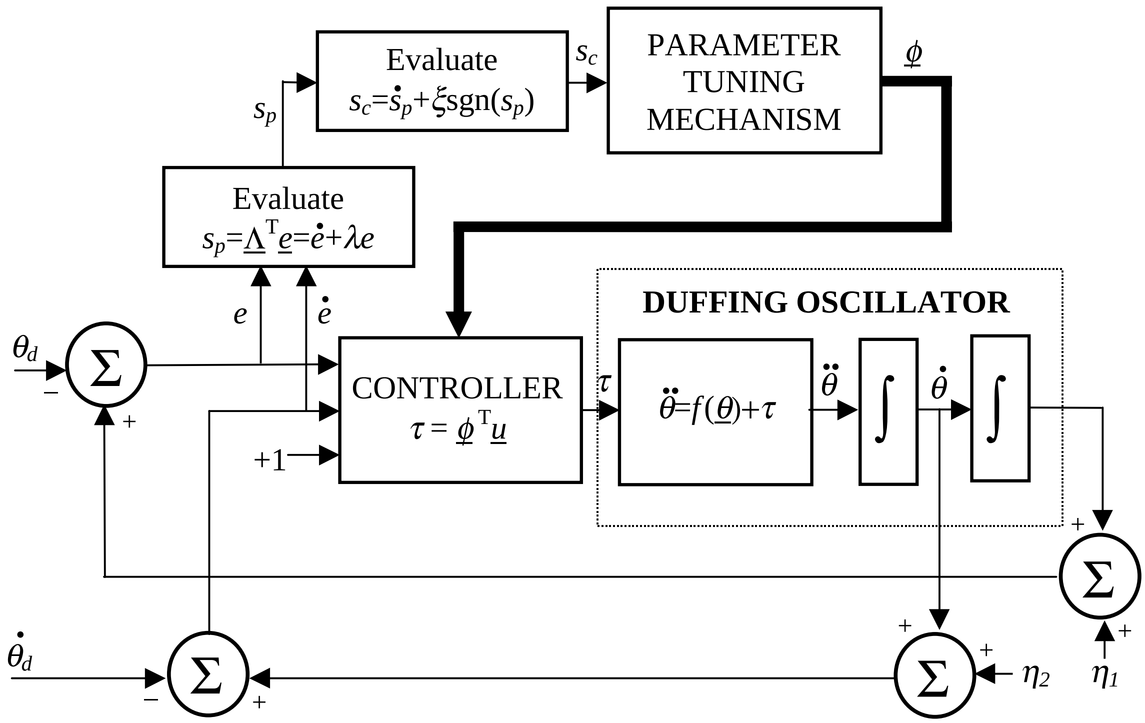

Figure 1. Control system structure

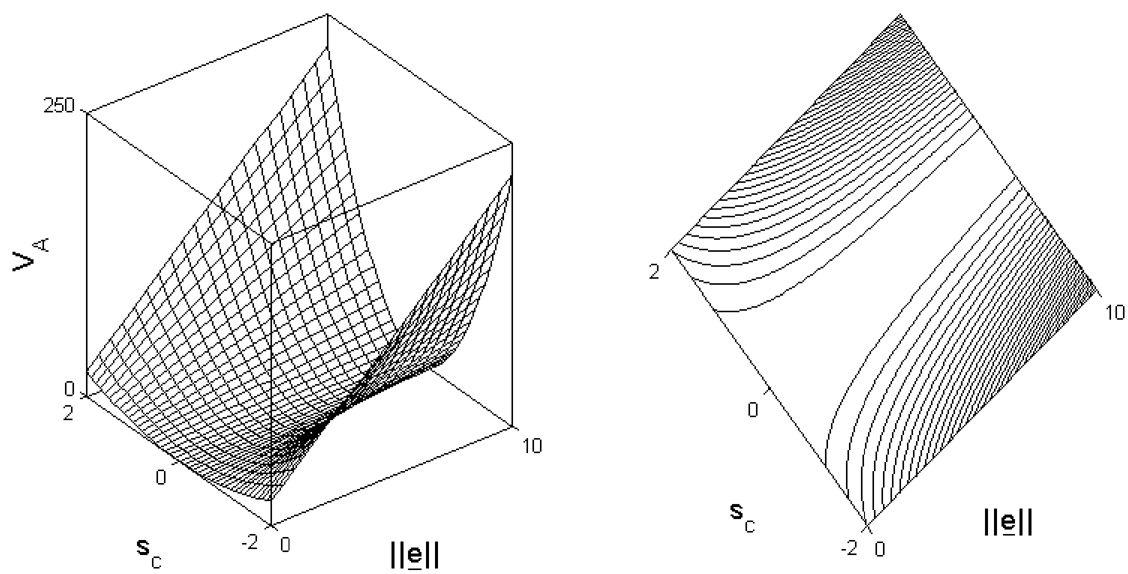

Figure 2. 3D Appearance and contour plot of V_A for μ=1, ρ=10

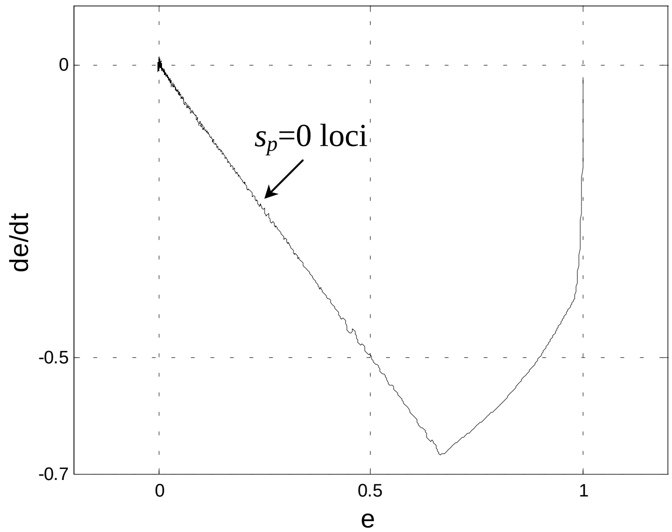

Figure 3. Phase space behavior

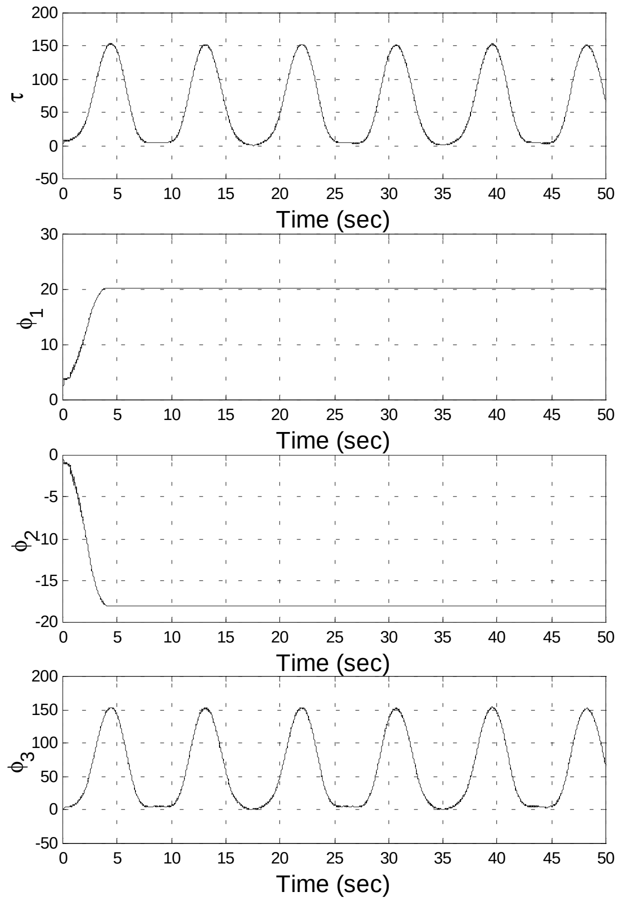

Figure 4. Applied control signal and the time evolution of the controller parameters